Arkhipov-Bezrukavnikov

Constructible side

G connected reductive split group over k. some coeff field
$F = k((t))$, $\mathcal{O} = k[[t]]$. $I \subset G_{\mathcal{O}}$ Iwahori group.
P_I, D_I : I-equivariant perverse / derived cat on $\mathcal{Fl} = G_F / I$.
H = affine Hecke algebra. H_f = finite Hecke algebra.
$sgn : H_f \to \mathbb{Z}[v]$, $H_s \mapsto -v$.
$M_{asph} = sgn \otimes_{H_f} H$ right H-module.
$W = X^\vee \rtimes W_f$ extended Hecke algebra.
${}^fW \subset W$: representatives of $W_f \backslash W$ of minimal length.
${}^fP_I$: categorification of M_{asph} : $P_I / \langle IC_w : w \notin {}^fW \rangle$.

Iwahori-Whittaker category

I^- opposite Iwahori, I_u^- pro-unipotent. $w \in W$, $\mathcal{Fl}^w$, I_u^- orbit on $\dot{w} \in \mathcal{Fl}$.
$\chi : I_u^- \to G_a$ character, $\mathcal{L}_\chi = \chi^* AS$ character sheaf.
P_{IW}, D_{IW} : $\mathcal{L}_\chi$-equivariant sheaf on $\mathcal{Fl}$.
$w \in {}^fW \Leftrightarrow stab_{I_u^-}(wI) \subseteq \ker(\chi) \Leftrightarrow \exists\ \mathcal{L}_\chi$-equiv on $\mathcal{Fl}^w$.
$\Delta_w = (\mathcal{Fl}^w \hookrightarrow \mathcal{Fl})_! \chi_w^* AS[\dim \mathcal{Fl}^w]$ $\nabla_w = (\mathcal{Fl}^w \hookrightarrow \mathcal{Fl})_* \chi_w^* AS[\dim \mathcal{Fl}^w]$
$w \in W$, $\mathcal{Fl}_w$: I-orbit on $\dot{w} \in \mathcal{Fl}$. $j_{w!} = j_{w!} \underline{\mathbb{Q}}[\dim \mathcal{Fl}_w]$, $j_{w*} = j_{w*} \underline{\mathbb{Q}}[\dim \mathcal{Fl}_w]$

Coherent side

$G^\vee$ Langlands dual group of G. $T^\vee \subset B^\vee \subset G^\vee$.
$\mathcal{B} = G^\vee / B^\vee$, $\mathcal{N} \subset \mathfrak{g}^\vee$ nil cone, $\tilde{\mathcal{N}} = T^*\mathcal{B}$ resolution $\circlearrowleft G^\vee$

Thm (AB 09) $D^{G^\vee}(\tilde{\mathcal{N}}) \longrightarrow D_I \to {}^fD_I \xrightarrow{\Delta_0 * -} D_{IW}$

is equivalence of category.

Central sheaf

$H_{sph} = Z(H)$ spherical Hecke algebra
$S : Rep(G^\vee) \to P_{G_{\mathcal{O}}}(Gr)$ Satake isomorphism.
[Gaitsgory] central functor $Z : Rep(G^\vee) \to P_I$

$\widetilde{Fl}$ on $\mathbb{A}^1$, restrict to $\mathbb{A}^1 \setminus \{0\}$ is $Gr \times G/B$,
restrict to $\{0\}$ is Fl.

$Z(S) = \Psi(S \boxtimes 1_{G/B}) \in P_I$ carries a log-monodromy $\mathcal{M}$.

moreover, $Z(S) * T \in P_I$ (use another scheme and nearby cycle)

Wakimoto sheaves

$\lambda \in X_+^\vee$, $J_\lambda = j_{\lambda *}$, $\lambda \in -X_+^\vee$, $J_\lambda = j_{\lambda !}$, $J_{\lambda_1+\lambda_2} = J_{\lambda_1} * J_{\lambda_2}$.

Claim: $J_\lambda \in P_I$ using the exatness of affine morphism

Thm $Z(V)$ has unique filtration for a totally ordered $X^\vee$,
whose subquotients are Wakimoto sheaves.

(uniqueness: $\mathrm{Hom}^\bullet(J_\lambda, J_\mu) = 0$ unless $\lambda \geq \mu$, $\mathrm{Hom}^\bullet(J_\lambda, J_\lambda) = \Lambda$)

Existence: a) convolution exact object has a filt of $J_\lambda * j_{w*}$,
b) if central, then has a filt of J_λ.

Let $\mathcal{A}$ denote the subcat whose objects has a filt of J_λ.
then $gr\mathcal{A} = Rep(T^\vee)$, $Rep(G^\vee) \xrightarrow{Z} \mathcal{A} \xrightarrow{gr} gr\mathcal{A}$ is $res^{G^\vee}_{T^\vee}$.

In particular, $Z_\lambda = Z(V_\lambda)$ has a head J_λ for $\lambda \in X_+^\vee$.

Let $b_\lambda : Z_\lambda \to J_\lambda$ be the map from canonical iso $j_\lambda^* Z_\lambda = \Lambda[\ell(\lambda)]$.

We have $b_\lambda \circ \mathcal{M}_{Z_\lambda} = 0$ since $\mathcal{M}_{Z_\lambda}$ is nilpotent. $\ell(\lambda) = \langle 2\rho, \lambda\rangle = \dim Fl_\lambda$

Construction of the functor

Some definition

$\mathcal{O}(G^\vee) = \bigoplus_{\lambda \in X_+^\vee} V_\lambda \otimes V_\lambda^\vee$, $\mathcal{O}(G^\vee/U^\vee) = \mathcal{O}(G^\vee)^{U^\vee} = \bigoplus_{\lambda \in X_+^\vee} V_\lambda$.

$\overline{G^\vee/U^\vee} = \mathrm{Spec}\, \mathcal{O}(G^\vee/U^\vee)$ the affine closure.

$\widetilde{\mathcal{N}} \subset G^\vee/B^\vee \times \mathfrak{g}^\vee$ pull back to $\widehat{\mathcal{N}} \subset G^\vee/U^\vee \times \mathfrak{g}^\vee$. $a : \mathfrak{g}^\vee \to \mathfrak{g}^\vee/\mathfrak{u}_p^\vee$

a vector field v_{taut} on $G^\vee/U^\vee \times \mathfrak{g}^\vee$ by $(a(x)(p), 0)$ on (p, x).

i.e., the derivation on $\mathcal{O}(\overline{G^\vee/U^\vee} \times \mathfrak{g}^\vee)$ by

$\bigoplus_{\lambda \in X_+^\vee} V_\lambda \longrightarrow \bigoplus_{\lambda \in X_+^\vee} V_\lambda \otimes \mathcal{O}(\mathfrak{g}^\vee)$ extending to $\mathcal{O}(\mathfrak{g}^\vee)$-linear.

the former is tautological derivation $V_\lambda \to V_\lambda \otimes \mathcal{O}(\mathfrak{g}^\vee)$.

Let $\widehat{\mathcal{N}}_{aff}$ be the zero of this vector field.

It's clear from definition $\widehat{\mathcal{N}}_{aff} \cap G^\vee/U^\vee \times \mathfrak{g}^\vee = \widehat{\mathcal{N}}$.

$\mathrm{Coh}^{G^\vee\times T^\vee}_{\mathrm{free}}(\overline{G^\vee/U^\vee}\times\mathfrak{g}^\vee)$

On $\mathrm{Coh}^{G^\vee\times T^\vee}_{\mathrm{free}}(\overline{G^\vee/U^\vee})$ a morphism $B_\lambda : V_\lambda\otimes\mathcal{O}\to\mathcal{O}(\lambda)$ induced by $m_{\lambda,\mu}: V_\lambda\otimes V_\mu\to V_{\lambda+\mu}$. shift the T action

It satisfies the Plücker relation

$$B_\lambda\otimes B_\mu = B_{\lambda+\mu}\circ(m_{\lambda,\mu}\otimes \mathrm{id}_{\mathcal{O}})$$

For every $V\in\mathrm{Rep}(G^\vee)$, there is a tautological map $V\to V\otimes\mathcal{O}(\mathfrak{g}^\vee)$, induce the $\mathcal{O}(\mathfrak{g}^\vee)$-linear map $N^{\mathrm{taut}}_V: V\otimes\mathcal{O}(\mathfrak{g}^\vee)\to V\otimes\mathcal{O}(\mathfrak{g}^\vee)$.

Lemmas a) Let A be a commutative algebra with $G^\vee$ action, all tensor endomorphisms of $\mathrm{Rep}(G^\vee)\to\mathrm{Coh}^{G^\vee}_{\mathrm{free}}(\mathrm{Spec}\,A): V\mapsto V\otimes A$ are represented by $\mathfrak{g}^\vee(\mathrm{Spec}\,A)$.

$N_{V_1\otimes V_2}=\mathrm{id}\otimes N_{V_2}+N_{V_1}\otimes\mathrm{id}$

By Tannakian formalism, $1+N$ is an element of $G^\vee(\mathrm{Spec}\,A[\varepsilon]/\varepsilon^2)$, whose image in $G^\vee(\mathrm{Spec}\,A)$ is identity.

b) Let A be a commutative algebra with $G^\vee\times T^\vee$ action. The set of morphisms $b_\lambda: V_\lambda\otimes A\to A(\lambda)$ in $\mathrm{Coh}^{G^\vee\times T^\vee}_{\mathrm{free}}(\mathrm{Spec}\,A)$ satisfying Plücker relations is represented by $\overline{G^\vee/U^\vee}(\mathrm{Spec}\,A)$.

$\bigoplus_{\lambda\in X^\vee_+} b_\lambda|_{V_\lambda\otimes 1}$ gives a map $\bigoplus_{\lambda\in X^\vee_+}V_\lambda\to A$,

Plücker relations show this is a ring homomorphism.

c) A with $G^\vee\times T^\vee$-action, N_V, b_λ morphisms in $\mathrm{Coh}^{G^\vee\times T^\vee}_{\mathrm{free}}(\mathrm{Spec}\,A)$, such that $b_\lambda\circ N_{V_\lambda}=0: V_\lambda\otimes A\to V_\lambda\otimes A\to A(\lambda)$ for all $\lambda\in X^\vee_+$, then it is represented by $\hat{\tilde{N}}_{\mathrm{aff}}(\mathrm{Spec}\,A)$.

From definition $\mathcal{O}(\overline{G^\vee/U^\vee}\times\mathfrak{g}^\vee)\xrightarrow[\text{corr. } N_{V_\lambda}]{d}\mathcal{O}(\overline{G^\vee/U^\vee}\times\mathfrak{g}^\vee)\xrightarrow[\text{corr. } b_\lambda]{} A$ is zero, hence it pass through the cokernel of d $=\mathcal{O}_{\hat{\tilde{N}}_{\mathrm{aff}}}$.

d) the above result extend to categorical sense:

the tensor endomorphism of $F:\mathrm{Rep}(G^\vee)\to\mathcal{C}$ extend to $\mathrm{Coh}^{G^\vee}_{\mathrm{free}}(\mathfrak{g}^\vee)\to\mathcal{C}$,

transformations $b_\lambda: F(V_\lambda)\to F(\lambda)$ of $F:\mathrm{Rep}(G^\vee\times T^\vee)\to\mathcal{C}$ extend to $\mathrm{Coh}^{G^\vee\times T^\vee}_{\mathrm{free}}(\overline{G^\vee/U^\vee})$,

$b_\lambda\circ N_{V_\lambda}=0$ for $F:\mathrm{Rep}(G^\vee\times T^\vee)\to\mathcal{C}$ extend to $\mathrm{Coh}^{G^\vee\times T^\vee}_{\mathrm{free}}(\hat{\tilde{N}}_{\mathrm{aff}})$.

Use the Ind-object $F(\underline{\mathcal{O}})$ to reduce to previous cases.

Hence we obtain a functor $\tilde{F}:\mathrm{Coh}^{G^\vee\times T^\vee}_{\mathrm{free}}(\hat{\tilde{N}}_{\mathrm{aff}})\to D$, by

$\widetilde{F}(V_\lambda \otimes \mathcal{O}) = Z_\lambda$, $\widetilde{F}(\mathcal{O}(\lambda)) = J_\lambda$, hence image in $\mathcal{A}$.

Factor to F

Let $\partial\widehat{\tilde{N}} = \widehat{\tilde{N}}_{aff} \setminus \widehat{\tilde{N}} = \widehat{\tilde{N}}_{aff} \cap (\overline{G^\vee/U^\vee} \setminus (G^\vee/U^\vee) \times \mathfrak{g}^\vee)$, it contains the supp of the cokernel of B_λ for $\lambda \in X^\vee_+$, and equals to it if λ is regular.

(Since $G^\vee/B^\vee$ is the multi-Proj scheme of $X^\vee$-graded algebra $\mathcal{O}(G^\vee/U^\vee)$.)

Furthermore, any sheaf set theoretically supported on $\overline{G^\vee/U^\vee} \setminus (G^\vee/U^\vee)$ is scheme-theoretically supported in cok B_λ for $\lambda \gg 0$.

Let K_λ denote the Koszul complex of B_λ, then $\widetilde{F}(K_\lambda) = 0$ since $gr\,\widetilde{F}(K_\lambda) \in gr\,\mathcal{A} = Rep(T^\vee)$ is zero.

For $\mathcal{F} \in K^b(Coh^{G^\vee \times T^\vee}_{free}(\widehat{\tilde{N}}_{aff}))$, whose cohomologies are set-support in $\partial\widehat{\tilde{N}}$, then it can be represented by a complex C, whose components are in $Coh_{\partial\widehat{\tilde{N}}}(\widehat{\tilde{N}}_{aff})$. Pick λ such that the supports of C are in cok B_λ, thus $B_\lambda \otimes id_C = 0$, showing $\mathcal{F}$ is a direct summand of $K_\lambda \otimes \mathcal{F}$.

Then show $K^b(Coh^{G^\vee \times T^\vee}_{free}(\widehat{\tilde{N}}_{aff})) / K^b_{\partial\widehat{\tilde{N}}} \hookrightarrow D^b(Coh^{G^\vee \times T^\vee}(\widehat{\tilde{N}}))$ fully faithful.

TO BE CONTINUED!